

Quasiopen and p -path open sets, and characterizations of quasicontinuity

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Abstract. In this paper we give various characterizations of quasiopen sets and quasicontinuous functions on metric spaces. For complete metric spaces equipped with a doubling measure supporting a p -Poincaré inequality we show that quasiopen and p -path open sets coincide. Under the same assumptions we show that all Newton-Sobolev functions on quasiopen sets are quasicontinuous.

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1. Introduction

When studying Sobolev spaces and potential theory on an open subset Ω of \mathbf{R}^n (or of a metric space), there are two natural Sobolev capacities one can consider. One defined using the global Sobolev norm and the other one using the Sobolev norm on Ω . When Ω is open, these two capacities are easily shown to have the same zero sets. We shall show that the same holds also if Ω is only quasiopen, i.e. open up to sets of arbitrarily small capacity, see Proposition 4.2 for the exact details.

To consider Sobolev spaces and capacities on nonopen sets is natural e.g. in fine potential theory. Such studies were pursued on quasiopen sets in \mathbf{R}^n by Kilpeläinen–Malý [22], Latvala [24] and Malý–Ziemer [25]. In the last two decades, several types of Sobolev spaces have been introduced on general metric spaces by e.g. Cheeger [11], Hajlasz [14] and Shanmugalingam [28]. Using this approach one just regards subsets as metric spaces in their own right, with the metric and the measure inherited from the underlying space. This makes it possible to define Sobolev type spaces on even more general subsets.

We shall use the Newtonian Sobolev spaces, which on open subsets of \mathbf{R}^n are known to coincide with the usual Sobolev spaces, see Theorem 4.5 in Shanmugalingam [28] and Theorem 7.13 in Hajlasz [14]. This equivalence is true also for open subsets of weighted \mathbf{R}^n with p -admissible weights, $p > 1$, see Propositions A.12 and A.13 in [4]. See also Heinonen–Koskela–Shanmugalingam–Tyson [19] for more on Newtonian spaces.

It is well known that every equivalence class of the classical Sobolev spaces contains better-than-usual, so-called *quasicontinuous*, representatives. In metric spaces this is only known to hold under certain assumptions. Roughly speaking, quasicontinuity means continuity outside sets of arbitrarily small capacity, see Definition 3.1. The Sobolev capacity plays a central role when defining quasicontinuity, and there are actually *two types* of quasicontinuity that one can consider on Ω , one for each of the two capacities mentioned above. As far as we know this subtle distinction has not been discussed in the literature. It is not difficult to show that these two notions of quasicontinuity are equivalent if Ω is open, and examples show that for general sets this is not true.

We shall show in Proposition 3.4 that the equivalence holds for functions defined on quasiopen sets. The proof is more involved than for open sets, but still rather elementary, and it holds in arbitrary metric spaces (only assuming that balls have finite measure). In Proposition 3.3 we obtain a similar equivalence for two notions of quasiopenness. These two results (and also Proposition 4.2) complement the restriction result from Björn–Björn [5, Proposition 3.5], stating that if $U \subset X$ is p -path open and measurable, then the minimal p -weak upper gradients with respect to X and U coincide in U . All these results show the equivalence between a global property and the corresponding property localized to a quasiopen or p -path open set.

It was shown by Shanmugalingam [29, Remark 3.5] that quasiopen sets in arbitrary metric spaces are *p -path open*, i.e. that p -almost every rectifiable curve meets such a set in a relatively open 1-dimensional set. We shall show that under the usual assumptions on the metric space (and in particular in \mathbf{R}^n), the converse implication is true as well. More precisely, we prove the following result.

Theorem 1.1. *Assume that the metric space X equipped with a doubling measure μ is complete and supports a p -Poincaré inequality. Then every p -path open set in X is quasiopen (and in particular measurable).*

Under the same assumptions it was recently shown by Björn–Björn–Latvala [7, Theorem 1.4] that a set is quasiopen if and only if it is a union of a finely open set and a set with zero capacity, generalizing a similar result from \mathbf{R}^n , see Adams–Lewis [1, Proposition 3]. Thus we now have two characterizations of quasiopen sets.

As a consequence of Theorem 1.1 we obtain the following characterization of quasicontinuous functions.

Theorem 1.2. *Assume that the metric space X equipped with a doubling measure μ is complete and supports a p -Poincaré inequality. Let $U \subset X$ be quasiopen.*

Then $u : U \rightarrow [-\infty, \infty]$ is quasicontinuous if and only if it is measurable and finite q.e., and $u \circ \gamma$ is continuous for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow U$.

In Proposition 3.4, we also provide a characterization of quasicontinuity using quasiopen sets in the spirit of Fuglede [13, Lemma 3.3]. In [7, Theorem 1.4] yet another characterization of quasicontinuity was given, this time in terms of fine continuity, see also [13, Lemma, p. 143].

Newtonian functions are defined more precisely than the usual Sobolev functions (in the sense that the classes of representatives are narrower), and under the

assumptions of Theorem 1.1, it was shown in Björn–Björn–Shanmugalingam [10] that *all* Newtonian functions on X and on open subsets of X are quasicontinuous. Moreover, the recent results in Ambrosio–Colombo–Di Marino [2] and Ambrosio–Gigli–Savaré [3] imply that the same holds if X is a complete doubling metric space and $1 < p < \infty$.

Using the characterization in Theorem 1.2 we can extend the quasicontinuity result from [10] to quasiopen sets as follows. See also Björn–Björn–Latvala [8] and Remark 4.5.

Theorem 1.3. *Assume that the metric space X equipped with a doubling measure μ is complete and supports a p -Poincaré inequality. Let $U \subset X$ be quasiopen. Then every function $u \in N_{\text{loc}}^{1,p}(U)$ is quasicontinuous.*

In the last section we weaken the assumptions in Theorem 1.1 and replace the doubling property and the Poincaré inequality by the requirement that bounded Newtonian functions are quasicontinuous, which is a much weaker assumption. In particular, we obtain the following result. In Section 5 we give a more general version using coanalytic sets.

Theorem 1.4. *Assume that X is complete, and that every bounded $u \in N^{1,p}(X)$ is quasicontinuous. Then every Borel p -path open set $U \subset X$ is quasiopen.*

We also prove a similar modification of Theorem 1.2, see Proposition 5.1. Our proofs of these generalized results (without doubling and Poincaré assumptions) are based on the following result which guarantees measurability of certain functions defined by their upper gradients. It may be of independent interest, and generalizes Corollary 1.10 from Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [20], where a similar measurability result was proved in the singleton case $X \setminus U = \{x_0\}$.

Proposition 1.5. *Assume that X is complete and separable, and that U is a coanalytic set. For every Borel function $\rho : X \rightarrow [0, \infty]$ define*

$$u_\rho(x) = \inf_{\gamma \in \Gamma_x} \int_\gamma \rho \, ds,$$

where Γ_x is the family of all rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$ (including constant curves) such that $\gamma(0) = x$ and $\gamma(l_\gamma) \in X \setminus U$. Then u_ρ is measurable.

The paper is organized as follows. In Section 2 we recall the necessary background on Newtonian spaces. Section 3 deals with the two notions of quasicontinuity. The results in that section are valid in arbitrary metric spaces. Theorems 1.1–1.3 are proved in Section 4. In Section 5 some partial generalizations of the results from Section 4 are proved without the Poincaré and doubling assumptions. We also formulate two open problems about Borel representatives of Newtonian functions.

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2. Notation and preliminaries

We assume throughout this paper that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $\mu(B) < \infty$ for all open balls $B \subset X$.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve γ is a curve with finite length l_γ . We will only consider curves which are compact and

rectifiable. Unless otherwise stated they will also be nonconstant and parameterized by arc length ds . We follow Heinonen and Koskela [18] in introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all nonconstant, compact and rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (2.1)$$

where we follow the convention that the left-hand side is ∞ whenever at least one of the terms therein is infinite. If g is a nonnegative measurable function on X and if (2.1) holds for p -almost every curve (see below), then g is a *p -weak upper gradient* of f .

Here we say that a property holds for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a p -weak upper gradient need not be a Borel function, it is only required to be measurable.

The p -weak upper gradients were introduced in Koskela–MacManus [23]. It was also shown there that if $g \in L^p(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \rightarrow g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has a *minimal p -weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p -weak upper gradient $g \in L^p(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [29] and Hajłasz [14]. The minimal p -weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p(X)$. Following Shanmugalingam [28], we define a version of Sobolev spaces on the metric measure space X .

Definition 2.2. Let for measurable f ,

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [28]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space. Nevertheless, we will still say that \tilde{u} is a *representative* of u if $\tilde{u} \sim u$.

Definition 2.3. The *Sobolev capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E .

The capacity is countably subadditive. A property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$ and v is everywhere defined, then $v \sim u$ if and only if $v = u$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [28] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e. In particular, \tilde{u} is a representative of u if and only if $\tilde{u} = u$ q.e.

We thus see that the equivalence classes in $N^{1,p}(X)/\sim$ are more narrowly defined than for the usual Sobolev spaces. In weighted \mathbf{R}^n (with a p -admissible weight and $p > 1$), $N^{1,p}(\mathbf{R}^n)/\sim$ coincides with the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [17, p. 96], see Björn–Björn [4, Appendix A.2].

For a measurable set $U \subset X$, the Newtonian space $N^{1,p}(U)$ is defined by considering $(U, d|_U, \mu|_U)$ as a metric space in its own right. It comes naturally with the intrinsic Sobolev capacity that we denote by C_p^U .

The measure μ is *doubling* if there exists a *doubling constant* $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$0 < \mu(2B) \leq C\mu(B) < \infty,$$

where $\delta B = B(x_0, \delta r)$. A metric space with a doubling measure is proper (i.e. such that closed and bounded subsets are compact) if and only if it is complete. See Heinonen [16] for more on doubling measures.

We will also need the following definition.

Definition 2.4. The space X supports a *p -Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions f on X and all $(p$ -weak) upper gradients g of f ,

$$\int_B |f - f_B| d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p},$$

where $f_B := \int_B f d\mu / \mu(B)$.

See Björn–Björn [4] or Heinonen–Koskela–Shanmugalingam–Tyson [19] for further discussion.

3. Quasicontinuity and quasiopen sets

We are now ready to define the two notions of quasicontinuity considered in this paper. We let C_p denote the Sobolev capacity taken with respect to the underlying space X and C_p^U will be the intrinsic Sobolev capacity taken with respect to U , i.e. with X in Definition 2.3 replaced by U .

Definition 3.1. Let $U \subset X$ be measurable. A function $u : U \rightarrow \overline{\mathbf{R}} := [-\infty, \infty]$ is C_p^U -*quasicontinuous* (resp. C_p -*quasicontinuous*) if for every $\varepsilon > 0$ there is a relatively open set $G \subset U$ such that $C_p^U(G) < \varepsilon$ (resp. an open set $G \subset X$ such that $C_p(G) < \varepsilon$) and such that $u|_{U \setminus G}$ is finite and continuous.

This distinction was tacitly suppressed in Björn–Björn–Shanmugalingam [10], Björn–Björn [4] and Björn–Björn–Latvala [6]–[8]. The first two deal only with quasicontinuous functions on open sets, and in this case the two definitions are relatively easily shown to be equivalent. In this note we show that the same equivalence holds for quasiopen U (as considered in [6]–[8]); the proof of this is more involved, although still rather elementary. The equivalence holds without any assumptions on the metric space other than that the measure of balls should be finite.

Definition 3.2. A set $U \subset X$ is *quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $G \cup U$ is open.

Quasiopen sets are measurable by Lemma 9.3 in Björn–Björn [5]. It is also quite easy to see that every $(C_p^U$ or $C_p)$ -quasicontinuous function on a measurable set (and thus in particular on a quasiopen set) is measurable.

The quasiopen sets do not (in general) form a topology. This is easily seen in unweighted \mathbf{R}^n with $p \leq n$ as all singleton sets are quasiopen, but not all sets are quasiopen. We shall prove the following characterizations of quasiopen sets.

Proposition 3.3. *Let $U \subset X$ be a quasiopen set and $V \subset U$. Then the following statements are equivalent:*

- (a) V is quasiopen (in X);
- (b) V is C_p -quasiopen in U , i.e. for every $\varepsilon > 0$ there is a relatively open set $G \subset U$ such that $C_p(G) < \varepsilon$ and $G \cup V$ is relatively open in U ;
- (c) V is C_p^U -quasiopen in U , i.e. for every $\varepsilon > 0$ there is a relatively open set $G \subset U$ such that $C_p^U(G) < \varepsilon$ and $G \cup V$ is relatively open in U .

Even though the quasiopen sets do not (in general) form a topology, we still have the characterization (iii) below of quasicontinuity using quasiopen sets.

Proposition 3.4. *Let $u : U \rightarrow \overline{\mathbf{R}}$ be a function on a quasiopen set $U \subset X$. Then the following statements are equivalent:*

- (i) u is C_p^U -quasicontinuous;
- (ii) u is C_p -quasicontinuous.

Moreover, if X is locally compact then these statements are equivalent to the following statement:

- (iii) u is finite q.e. and the sets $U_\alpha := \{x \in U : u(x) > \alpha\}$ and $V_\alpha := \{x \in U : u(x) < \alpha\}$, $\alpha \in \mathbf{R}$, are quasiopen (in any of the equivalent senses (a), (b) and (c) of Proposition 3.3).

Remark 3.5. The local compactness assumption is only needed when showing that there is an open neighbourhood of $\{x : |u(x)| = \infty\}$ with small capacity when proving (iii) \Rightarrow (ii). In particular (i) \Leftrightarrow (ii) \Leftrightarrow (iii) holds without this assumption for real-valued functions $u : U \rightarrow \mathbf{R}$.

Example 3.6. The equivalence between the two types of quasicontinuity does not hold for arbitrary measurable subsets: To see this let e.g. $U = (\mathbf{R} \setminus \mathbf{Q})^2$ as a subset of (unweighted) \mathbf{R}^2 and $u = \chi_{[0, \sqrt{2}]^2}$. Then any relatively open set $G \subset U$ such that $u|_{U \setminus G}$ is continuous must contain at least one point on each line $\{(x, y) : x = t\}$ for $0 < t < \sqrt{2}$, $t \notin \mathbf{Q}$. It follows, by projection, that $C_p(G) \geq C_p([0, \sqrt{2}] \setminus \mathbf{Q} \times \{0\}) > 0$ and thus u is not C_p -quasicontinuous. On the other hand, let $E = U \cap (([0, \sqrt{2}] \times \{\sqrt{2}\}) \cup (\{\sqrt{2}\} \times [0, \sqrt{2}]))$. Then $u|_{U \setminus E}$ is continuous. Since $\mu(E) = 0$, the regularity of the measure shows that, for every $\varepsilon > 0$, there is a relatively open subset G of U such that $E \subset G$ and $\mu(G) < \varepsilon$. As there are no nonconstant curves in U , we have $C_p^U(G) = \mu(G)$, and thus u is C_p^U -quasicontinuous.

In Fuglede [13, Lemma 3.3], a characterization similar to (iii) was obtained for general capacities on topological spaces. The definitions of quasiopen sets and quasicontinuity therein differ however somewhat from ours. More precisely, in [13], a set V is called quasiopen if for every $\varepsilon > 0$ there exists an open set Ω such that the symmetric difference $(V \setminus \Omega) \cup (\Omega \setminus V)$ has capacity less than ε . If the capacity is outer then this notion is easily shown to be equivalent to our definition, but in general Fuglede's definition allows for more quasiopen sets.

Similarly, in the definition of quasicontinuity in [13], it is not in general required that the removed exceptional set G be open (though for outer capacities this can always be arranged) and continuity does not require finiteness. In other words, Fuglede's definition of quasicontinuity corresponds to the *weak quasicontinuity* considered (on open sets) in Björn–Björn [4, Section 5.2], less the requirement that continuous functions be finite. Fuglede's notion of quasicontinuity is in [13, Lemma 3.3] proved to be equivalent to the fact that the sets U_α and V_α , $\alpha \in \mathbf{R}$, in Proposition 3.4 are quasiopen (in the sense of [13]).

Since our definitions do not exactly agree with those in [13], and moreover we consider quasicontinuity and quasiopenness with respect to two different capacities (C_p and C_p^U) and two possible underlying spaces (X and U), we present here for the reader's convenience full proofs of these results.

The proof of Proposition 3.4 can be easily modified to show that on quasiopen sets weak quasicontinuity is the same with respect to C_p and C_p^U .

Proof of Proposition 3.3. (a) \Rightarrow (b) For every $\varepsilon > 0$ there exists an open $G \subset X$ such that $C_p(G) < \varepsilon$ and $\Omega := V \cup G$ is open. Then $\tilde{G} := G \cap U$ and $\tilde{\Omega} := \Omega \cap U$ are relatively open in U and $C_p(\tilde{G}) < \varepsilon$, i.e. (b) holds.

(b) \Rightarrow (c) This is straightforward, since C_p^U is majorized by C_p .

(c) \Rightarrow (a) We may assume that $\emptyset \neq U \neq X$. Let $\varepsilon > 0$ be arbitrary. Since U is quasiopen, there is an open set G such that $C_p(G) < \varepsilon$ and such that $\Omega := U \cup G \neq X$ is open. Hence there is $w \in N^{1,p}(X)$ such that $\chi_G \leq w \leq 1$ and $\|w\|_{N^{1,p}(X)}^p < \varepsilon$.

Using that V is C_p^U -quasiopen, we can find, for each $j = 1, 2, \dots$, a relatively open set $G_j \subset U$ such that $V \cup G_j$ is relatively open in U and $C_p^U(G_j) < \varepsilon_j := 2^{-j} j^{-p} \varepsilon$. There is thus $v_j \in N^{1,p}(U)$ such that $\chi_{G_j} \leq v_j \leq 1$ in U and $\|v_j\|_{N^{1,p}(U)}^p < \varepsilon_j$. Next, let $x_0 \in \Omega$,

$$\Omega_j = \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > 1/j \text{ and } d(x, x_0) < j\}, \quad j = 1, 2, \dots,$$

and $\eta_j(x) = (1 - j \text{dist}(x, \Omega_j))_+$. (If $\Omega_j = \emptyset$ we let $\eta_j \equiv 0$.) Define the functions

$$\varphi_k = \begin{cases} \min\{1 - w, \max_{1 \leq j \leq k} v_j \eta_j\} & \text{in } U, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \varphi = \lim_{k \rightarrow \infty} \varphi_k.$$

Then φ_k has bounded support and $\varphi_k \in N^{1,p}(U)$. Since

$$0 \leq \varphi_k \leq \min\{1 - w, \eta_k\} \in N_0^{1,p}(U) := \{f : f \in N^{1,p}(X) \text{ and } f = 0 \text{ outside } U\},$$

Lemma 2.37 in [4] implies that $\varphi_k \in N_0^{1,p}(U) \subset N^{1,p}(X)$. Next,

$$\int_X \varphi_k^p d\mu \leq \sum_{j=1}^k \int_X v_j^p d\mu < \sum_{j=1}^k \varepsilon_j < \varepsilon. \quad (3.1)$$

By Lemma 1.52 in [4], $g := \sup_k g_{\varphi_k}$ is a p -weak upper gradient of φ . For a.e. $x \in U$ we either have $g = g_{\varphi_k} = g_w$ or $g = g_{\varphi_k} = g_{v_j \eta_j} \leq g_{v_j} + j v_j$ for some j and k (see [4, Theorem 2.15 and Corollary 2.21]). Hence

$$\int_X g^p d\mu \leq \int_X g_w^p d\mu + \sum_{j=1}^{\infty} 2^{p-1} \int_U (g_{v_j}^p + j^p v_j^p) d\mu \leq \varepsilon + \sum_{j=1}^{\infty} 2^{p-1} j^p \varepsilon_j < \varepsilon + 2^p \varepsilon,$$

which together with (3.1) shows that $\|\varphi\|_{N^{1,p}(X)}^p \leq 2\varepsilon + 2^p \varepsilon$.

Next, set

$$H = G \cup \bigcup_{j=1}^{\infty} (G_j \cap \Omega_j),$$

which is open by the choice of G as $\bigcup_{j=1}^{\infty} (G_j \cap \Omega_j)$ is relatively open in U . Then $w + \varphi \geq \chi_H$ and hence

$$C_p(H) \leq \|w + \varphi\|_{N^{1,p}(X)}^p \leq 2^{p-1} (\|w\|_{N^{1,p}(X)}^p + \|\varphi\|_{N^{1,p}(X)}^p) \leq 2^{p-1} (3\varepsilon + 2^p \varepsilon).$$

It remains to show that $V \cup H$ is open in X . If $x \in V$, then $x \in \Omega_k$ for some k , and as $V \cup G_k$ is relatively open in U (and thus $G \cup V \cup G_k$ is open in X), we can find a ball B_x such that

$$x \in B_x \subset (G \cup V \cup G_k) \cap \Omega_k \subset V \cup G \cup (G_k \cap \Omega_k) \subset V \cup H.$$

For $x \in H$, we can instead choose $x \in B_x \subset H \subset V \cup H$. \square

Proof of Proposition 3.4. (ii) \Rightarrow (i) This is straightforward, since C_p^U is majorized by C_p .

(i) \Rightarrow (ii) We may assume that $\emptyset \neq U \neq X$. Let $\varepsilon > 0$ be arbitrary. For each $j = 1, 2, \dots$, there is a relatively open set $G_j \subset U$ such that $C_p^U(G_j) < \varepsilon_j := 2^{-j} j^{-p} \varepsilon$ and such that $u|_{U \setminus G_j}$ is finite and continuous in U .

Next construct the open sets G , H and Ω_k , $k = 1, 2, \dots$, as in the proof of Proposition 3.3, (c) \Rightarrow (a). Then $C_p(H) \leq 2^{p-1}(3\varepsilon + 2^p \varepsilon)$. If $x \in U \setminus H$, then there is k such that $x \in \Omega_k$. Since $u|_{U \setminus G_k}$ is finite and continuous at x , it thus follows that $u|_{U \setminus H}$ is finite and continuous at x . Hence $u|_{U \setminus H}$ is finite and continuous, and u is C_p -quasicontinuous.

Now assume that X is locally compact.

(ii) \Rightarrow (iii) Let $\varepsilon > 0$. Since U is quasiopen we can find an open set G such that $G \cup U$ is open and $C_p(G) < \varepsilon$. As u is C_p -quasicontinuous, there is an open set $H \subset X$ such that $C_p(H) < \varepsilon$ and $u|_{U \setminus H}$ is continuous. Thus, for every $\alpha \in \mathbf{R}$, $U_\alpha \setminus H$ is relatively open in $U \setminus H$. Hence $U_\alpha \cup (G \cup H)$ is open and $C_p(G \cup H) < 2\varepsilon$, showing that U_α is quasiopen. That V_α is quasiopen follows similarly, whereas u is finite q.e. by definition.

(iii) \Rightarrow (ii) Let $\varepsilon > 0$ and $E = \{x \in U : |u(x)| = \infty\}$. By assumption, $C_p(E) = 0$ and thus there is an open set $H \supset E$ such that $C_p(H) < \varepsilon$, by Proposition 1.4 in Björn–Björn–Shanmugalingam [10] and Proposition 4.7 in Björn–Björn–Lehrbäck [9]. Next let $\{q_j\}_{j=1}^\infty$ be an enumeration of \mathbf{Q} . By assumption there are open G_j such that $U_{q_j} \cup G_j$ and $V_{q_j} \cup G_j$ are open and $C_p(G_j) < 2^{-j}\varepsilon$. Let $G = H \cup \bigcup_{j=1}^\infty G_j$, which is open and such that $C_p(G) < 2\varepsilon$. Moreover, $U_{q_j} \setminus G$ is relatively open in $U \setminus G$. For $\alpha \in \mathbf{R}$ it follows that

$$U_\alpha \setminus G = \bigcup_{\mathbf{Q} \ni q > \alpha} (U_q \setminus G)$$

is relatively open in $U \setminus G$. Similarly $V_\alpha \setminus G$ is relatively open in $U \setminus G$, and thus $u|_{U \setminus G}$ is finite and continuous. \square

Theorem 1.1 shows that under certain assumptions on X , p -path open sets are quasiopen, and thus Propositions 3.3 and 3.4 hold for p -path open sets in that case. In general, p -path open sets need not be measurable, see Example 4.6 below. It would therefore be interesting to know if the conclusions of Propositions 3.3 and 3.4 hold for measurable p -path open sets U (or even measurable p -path almost open sets U , see Björn–Björn [5]) without additional assumptions on X . In Section 5 some partial results are obtained for p -path open sets which are Borel. Note that in the situation described in Example 4.6 below, the conclusions of Propositions 3.3 and 3.4 do hold for measurable p -path open sets.

4. p -path open and quasiopen sets

Definition 4.1. A set $G \subset X$ is *p -path open* (in X) if for p -almost every curve $\gamma : [0, l_\gamma] \rightarrow X$, the set $\gamma^{-1}(G)$ is (relatively) open in $[0, l_\gamma]$.

The arguments proving Propositions 3.3 and 3.4 can also be used to show that C_p and C_p^U have the same zero sets for quasiopen U . Using a different approach we can obtain this more generally for p -path open sets.

Proposition 4.2. Let U be a measurable p -path open set and $E \subset U$. Then $C_p(E) = 0$ if and only if $C_p^U(E) = 0$.

For open U this is well-known, see e.g. Lemma 2.24 in Björn–Björn [4].

Proof. Assume that $C_p^U(E) = 0$. Proposition 1.48 in [4], applied with U as the underlying space, implies that $\mu(E) = 0$ and that p -almost every curve in U avoids E . Since U is p -path open, it follows that also p -almost every curve in X avoids E (see Lemma 3.3 in Björn–Björn–Latvala [8]). Thus, by Proposition 1.48 in [4] again (this time with respect to X), $C_p(E) = 0$.

The converse implication is trivial. \square

The p -path open sets were introduced by Shanmugalingam [29, Remark 3.5]. It was also shown there that every quasiopen set is p -path open. We are now going to prove Theorem 1.1 which says that the converse is true under suitable assumptions on X . In particular, this holds in \mathbf{R}^n .

Proof of Theorem 1.1. Let $U \subset X$ be p -path open. Then the family Γ of curves γ in X , for which $\gamma^{-1}(U)$ is not relatively open, has zero p -modulus, i.e. there exists $\rho \in L^p(X)$ such that $\int_\gamma \rho ds = \infty$ for every $\gamma \in \Gamma$.

Assume to start with that U is bounded and let B be a ball containing U . Define, for $x \in X$,

$$u(x) = \min \left\{ 1, \inf_\gamma \int_\gamma (\rho + \chi_B) ds \right\}, \quad (4.1)$$

where χ_B is the characteristic function of B and the infimum is taken over all rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$ (including constant curves) such that $\gamma(0) = x$ and $\gamma(l_\gamma) \in X \setminus U$. Then $u = 0$ in $X \setminus U$ and Lemma 3.1 in Björn–Björn–Shanmugalingam [10] (or [4, Lemma 5.25]) shows that u has $\rho + \chi_B$ as an upper gradient. Since the measure μ is doubling and X is complete and supports a p -Poincaré inequality, we can conclude from Theorem 1.11 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [20] that u is measurable. As U is assumed to be bounded and $\rho \in L^p(X)$, it follows that $u \in N^{1,p}(X)$.

We claim that $u > 0$ in U , i.e. that $U = \{x \in X : u(x) > 0\}$ is a superlevel set of a Newtonian function. The assumptions on X guarantee that u is quasicontinuous (by [10, Theorem 1.1] or [4, Theorem 5.29]), which then directly implies that U is quasiopen, see Proposition 3.4.

To prove the claim, let $x \in U$ and assume for a contradiction that $u(x) = 0$. Then there exist curves γ_j connecting x to $X \setminus U$ such that

$$\int_{\gamma_j} (\rho + \chi_B) ds \leq 2^{-j}, \quad j = 1, 2, \dots \quad (4.2)$$

In particular, $l_{\gamma_j} \leq 2^{-j}$ for all $j = 1, 2, \dots$.

We define a curve $\tilde{\gamma}$ as a recursive concatenation of all γ_j and their reversed reparameterizations as follows. Let $L_0 = 0$,

$$L_j = 2 \sum_{i=1}^j l_{\gamma_i} \leq 2 \sum_{i=1}^\infty l_{\gamma_i} =: L \leq 2 \sum_{j=1}^\infty 2^{-j} = 2$$

and

$$\tilde{\gamma}(t) = \begin{cases} \gamma_j(t - L_{j-1}), & \text{if } L_{j-1} \leq t \leq L_{j-1} + l_{\gamma_j}, \quad j = 1, 2, \dots, \\ \gamma_j(L_j - t), & \text{if } L_{j-1} + l_{\gamma_j} \leq t \leq L_j, \quad j = 1, 2, \dots \end{cases}$$

Then $\tilde{\gamma} : [0, L] \rightarrow X$, $\tilde{\gamma}(L) = x$ and $\tilde{\gamma}(L_j + l_{\gamma_{j+1}}) \in X \setminus U$, $j = 1, 2, \dots$. Since $x \in U$ and $L_j + l_{\gamma_{j+1}} \rightarrow L$, as $j \rightarrow \infty$, this shows that $\tilde{\gamma}^{-1}(U)$ is not relatively open in $[0, L]$ and hence $\tilde{\gamma} \in \Gamma$. But (4.2) implies that

$$\int_{\tilde{\gamma}} \rho ds \leq 2 \sum_{j=1}^\infty \int_{\gamma_j} (\rho + \chi_B) ds \leq 2 \sum_{j=1}^\infty 2^{-j} = 2,$$

contradicting the choice of ρ . We can therefore conclude that $u(x) > 0$ for all $x \in U$, which finishes the proof for bounded U .

If U is unbounded, then the above argument shows that $U \cap B_j$ is quasiopen, $j = 1, 2, \dots$, where $\{B_j\}_{j=1}^\infty$ is a countable cover of X by (open) balls. In particular, given $\varepsilon > 0$, there exists for every $j = 1, 2, \dots$ an open set G_j such that $C_p(G_j) < 2^{-j}\varepsilon$ and $(U \cap B_j) \cup G_j$ is open. Setting $G = \bigcup_{j=1}^\infty G_j$, we see that $C_p(G) < \varepsilon$ and $U \cup G = \bigcup_{j=1}^\infty ((U \cap B_j) \cup G_j)$ is open, which concludes the proof. \square

We now turn to Theorem 1.2 and restate it in a more general form.

Proposition 4.3. *Assume that X is locally compact and that every measurable p -path open set in X is quasiopen. Let $u : U \rightarrow \overline{\mathbf{R}}$ be a function on a quasiopen set U . Then the following are equivalent:*

- (a) *u is quasicontinuous (with respect to C_p or C_p^U);*
- (b) *u is measurable and finite q.e., and $u \circ \gamma$ is continuous (on the set where it is defined) for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow X$;*
- (c) *u is measurable and finite q.e., and $u \circ \gamma$ is continuous for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow U$.*

The assumptions of Proposition 4.3 are guaranteed by Theorem 1.1, but there are also other situations when they hold, see e.g. Example 4.6 below and Example 5.6 in [4].

As seen from the proof below and Remark 3.5, no assumptions on X are needed for the implications (a) \Rightarrow (b) \Leftrightarrow (c) in Proposition 4.3. Also, if we know that every p -path open set is quasiopen (and thus measurable), then the measurability assumption for u can be dropped, cf. the proof of Theorem 1.1 where measurability of u follows from Theorem 1.11 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [20].

For real-valued u , the assumption of local compactness here and in Corollary 4.4 below can be omitted, see Remark 3.5.

Proof. (a) \Rightarrow (b) Assume that $u : U \rightarrow \overline{\mathbf{R}}$ is quasicontinuous. Proposition 3.4 shows that it is finite q.e. and the sets $U_\alpha := \{x \in U : u(x) > \alpha\}$ and $V_\alpha := \{x \in U : u(x) < \alpha\}$, $\alpha \in \mathbf{R}$, are quasiopen. Remark 3.5 in Shanmugalingam [29] implies that U , U_α and V_α , $\alpha \in \mathbf{R}$, are p -path open. Hence, for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow X$, the set $I_\gamma = \gamma^{-1}(U)$ is a relatively open subset of $[0, l_\gamma]$, and so are the level sets

$$\gamma^{-1}(U_\alpha) = \{t \in I_\gamma : (u \circ \gamma)(t) > \alpha\} \quad \text{and} \quad \gamma^{-1}(V_\alpha) = \{t \in I_\gamma : (u \circ \gamma)(t) < \alpha\}, \quad (4.3)$$

for all $\alpha \in \mathbf{Q}$. This implies that $u \circ \gamma : I_\gamma \rightarrow \overline{\mathbf{R}}$ is continuous for p -a.e. curve γ .

(b) \Rightarrow (c) This is trivial.

(c) \Rightarrow (b) Let Γ be the family of exceptional curves $\gamma : [0, l_\gamma] \rightarrow U$ for which $u \circ \gamma$ is not continuous. As U is quasiopen, and thus p -path open by Remark 3.5 in [29], $\gamma^{-1}(U)$ is open for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow X$. By Lemma 1.34 in [4], p -a.e. such curve γ does not have a subcurve in Γ . For such a γ , the composition $u \circ \gamma$ is continuous on the relatively open subset of $[0, l_\gamma]$ where it is defined.

(b) \Rightarrow (a) Since U is quasiopen, and thus p -path open by Remark 3.5 in [29], for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow X$, the set $I_\gamma = \gamma^{-1}(U)$ is a relatively open subset of $[0, l_\gamma]$ and $u \circ \gamma$ is continuous on I_γ . For such γ , and all $\alpha \in \mathbf{R}$, the level sets in (4.3) are relatively open in I_γ , and thus in $[0, l_\gamma]$. It follows that the level sets $U_\alpha := \{x \in U : u(x) > \alpha\}$ and $V_\alpha := \{x \in U : u(x) < \alpha\}$, $\alpha \in \mathbf{R}$, are p -path open and measurable (as u is measurable), and thus quasiopen by the assumption. Proposition 3.4 now concludes the proof. \square

Proof of Theorem 1.2. Theorem 1.1 guarantees that the assumptions of Proposition 4.3 are satisfied, from which the result follows. \square

As a consequence of the characterization in Proposition 4.3 we obtain the following generalization of Theorem 1.3 and partial converse of Theorem 1.4.

Corollary 4.4. *Assume that X is locally compact and that every measurable p -path open set in X is quasiopen. Let $U \subset X$ be quasiopen. Then all $u \in N_{\text{loc}}^{1,p}(U)$ are quasicontinuous.*

Note that in particular we may let $U = X$.

Proof. Since every Newtonian function is measurable, finite q.e. (by [4, Proposition 1.30]) and absolutely continuous on p -a.e. curve, by Shanmugalingam [28, Proposition 3.1] (or [4, Theorem 1.56]), this follows directly from Proposition 4.3. \square

Proof of Theorem 1.3. This follows directly from Theorem 1.1 and Corollary 4.4. \square

Remark 4.5. Theorem 1.3 was recently obtained by Björn–Björn–Latvala [8]. The proof given here is very different from that in [8] and appears as an immediate corollary of other results. In particular, it does not use the fine topology and the quasi-Lindelöf principle, whose proof in [8] relies on the vector-valued, so-called Cheeger, differentiable structure.

The assumptions of our Corollary 4.4 are weaker than those of the version of Theorem 1.3 in [8]. On the other hand, in [8], quasicontinuity is deduced for a larger local space than $N_{\text{loc}}^{1,p}(U)$. See [8] for more details and the precise definitions of the local spaces considered therein.

The following example shows that p -path open sets need not be quasiopen in general.

Example 4.6. Assume that there are no nonconstant rectifiable curves in X . For example, consider \mathbf{R} with the snowflaked metric $d(x, y) = |x - y|^\alpha$, $0 < \alpha < 1$, and the Lebesgue measure.

Then every set is p -path open, while Lemma 9.3 in Björn–Björn [5] shows that quasiopen sets must be measurable. Any nonmeasurable set $U \subset X$ is thus p -path open but cannot be quasiopen. Indeed, the function u constructed in the proof of Theorem 1.1 is χ_U , which has zero as an upper gradient, but it is not measurable and thus not in $N^{1,p}(X)$.

Note that in this case the zero function is an upper gradient of every function and thus $N^{1,p}(X) = L^p(X)$ and C_p is the extension of the measure μ to all subsets of X as an outer measure. It thus follows that the quasiopen sets are just the measurable sets, and hence every measurable p -path open set is quasiopen. Thus Corollary 4.4 applies in this case, but this already follows (in this particular case) from Luzin’s theorem.

The same argument applies also if the family of nonconstant rectifiable curves has zero p -modulus. (In this case “upper gradient” should be replaced by “ p -weak upper gradient” above.) In fact, this assumption is equivalent to the equality $N^{1,p}(X) = L^p(X)$ as sets of functions, see L. Malý [26, Lemma 2.5].

5. Outside the realm of a Poincaré inequality

The doubling condition and the Poincaré inequality are standard assumptions in analysis on metric spaces. In particular, they guarantee that Lipschitz functions are

dense in $N^{1,p}(X)$, which in turn (together with completeness) implies the quasicontinuity of Newtonian functions used in the proof of Theorem 1.1. On the other hand, there are plenty of spaces where the Poincaré inequality fails or the doubling condition is violated, but where Newtonian functions are quasicontinuous. Therefore, in this section we relax the assumptions of Theorem 1.1 and obtain Theorem 1.4.

We postpone the proof of Theorem 1.4 to the end of this section. Meanwhile, we formulate some consequences of it. First, we have the following characterization of quasicontinuity among Borel functions, analogous to Proposition 4.3. Note that the assumption of quasiopenness of p -path open sets can be deduced using Theorem 1.4, or its generalization Theorem 5.7 below, under the assumptions therein.

Proposition 5.1. *Assume that X is locally compact and that every Borel p -path open set in X is quasiopen. Let $u : U \rightarrow \overline{\mathbf{R}}$ be a Borel function on a quasiopen set U . Then the following are equivalent:*

- (a) *u is quasicontinuous (with respect to C_p or C_p^U);*
- (b) *u is finite q.e. and $u \circ \gamma$ is continuous (on the set where it is defined) for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow X$;*
- (c) *u is finite q.e. and $u \circ \gamma$ is continuous for p -a.e. curve $\gamma : [0, l_\gamma] \rightarrow U$.*

Proof. As mentioned before the proof of Proposition 4.3, the proof of (a) \Rightarrow (b) \Leftrightarrow (c) therein holds without any assumptions on X and therefore applies also here.

(b) \Rightarrow (a) Replace the word “measurable” by “Borel” twice in the proof of this implication in the proof of Proposition 4.3. \square

Corollary 5.2. *Assume that X is complete and locally compact. Consider the following statements:*

- (a) *every bounded $u \in N^{1,p}(X)$ is quasicontinuous;*
- (b) *every $u \in N_{\text{loc}}^{1,p}(X)$ is quasicontinuous;*
- (c) *every Borel p -path open set in X is quasiopen;*
- (d) *if U is quasiopen, then every Borel function $u \in N_{\text{loc}}^{1,p}(U)$ is quasicontinuous.*

Then (a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d).

Moreover, if every bounded function $u \in N^{1,p}(X)$ has a Borel representative $\tilde{u} \in N^{1,p}(X)$, then all the statements are equivalent.

Recall that \tilde{u} is termed a representative of u if both functions belong to the same equivalence class in $N^{1,p}(X)/\sim$ (or, equivalently, if $\tilde{u} = u$ q.e., see Section 2).

Proof. (a) \Rightarrow (b) Let $u \in N_{\text{loc}}^{1,p}(X)$. Fix $x_0 \in X$ and let $\eta_j \in \text{Lip}(X)$ be such that $\eta_j = 1$ on $B(x_0, j)$ and $\eta_j = 0$ outside $B(x_0, 2j)$. Then $u_j := u\eta_j \in N^{1,p}(X)$ and also $\arctan u_j \in N^{1,p}(X)$. By assumption $\arctan u_j$ is quasicontinuous. As u_j is finite q.e., Proposition 1.4 in [10] and Proposition 4.7 in [9] show that also u_j is quasicontinuous in X . Hence u is quasicontinuous in $B(x_0, j)$ for each j , and it follows from Lemma 5.18 in [4] that u is quasicontinuous in X .

(b) \Rightarrow (a) This is trivial.

(a) \Rightarrow (c) This is Theorem 1.4.

(c) \Rightarrow (d) Let $u \in N_{\text{loc}}^{1,p}(U)$ be a Borel function. Then it is finite q.e. and absolutely continuous on p -a.e. curve (see Shanmugalingam [28, Proposition 3.1] or Proposition 1.30 and Theorem 1.56 in [4]). The quasicontinuity of u then follows from Proposition 5.1.

Finally, assume that every bounded function $u \in N^{1,p}(X)$ has a Borel representative $\tilde{u} \in N^{1,p}(X)$ and that (d) holds with $U = X$. Let $u \in N^{1,p}(X)$ be bounded and $\tilde{u} \in N^{1,p}(X)$ be a Borel representative of u . By assumption \tilde{u} is quasicontinuous. As X is locally compact, Proposition 1.4 in Björn–Björn–Shanmugalingam [10] (or [4, Proposition 5.27]) and Proposition 4.7 in Björn–Björn–Lehrbäck [9] show that also u is quasicontinuous. Hence (a) holds. \square

The following open problems are natural in view of Corollary 5.2 and the comment after it.

Open problem 5.3. Does every (bounded) $u \in N^{1,p}(X)$ have a Borel representative?

Note that if every bounded $u \in N^{1,p}(X)$ has a Borel representative, then also every unbounded $v \in N^{1,p}(X)$ has a Borel representative. Indeed, if $u = \arctan v$ has a Borel representative \tilde{u} (which we may require to have values in $(-\pi/2, \pi/2)$), then $\tan \tilde{u}$ becomes a Borel representative of v .

Since any quasiopen set can be written as a union of a Borel set and a set of capacity zero (cf. [7, Lemma 9.5] Proposition 3.4, shows that the answer to Open problem 5.3 is positive for a particular space X if all Newtonian functions on X are quasicontinuous.

Open problem 5.4. Can “Borel” in Theorem 1.4 be replaced by “measurable”? Example 4.6 shows that it cannot be omitted altogether.

It would follow that a version of Corollary 5.2 with “Borel” replaced by “measurable” would also be possible, and since all Newtonian functions are measurable, the equivalence of (a)–(d) therein would follow in that case.

We now proceed to the proof of Theorem 1.4. For this, we will need Proposition 1.5 about measurability of the function u in (4.1), which does not rely on any Poincaré inequality.

Recall that a subset of a complete separable metric space is *analytic* (or *Suslin*) if it is a continuous image of a complete separable metric space, see e.g. Kechris [21, Definitions 3.1 and 14.1]. By Theorem 14.11 (Suslin’s theorem) therein, every Borel subset of a complete separable space is analytic. (In fact, it shows that Borel sets are exactly those analytic sets which are also *coanalytic*, i.e. whose complements are analytic.)

Proposition 14.4 in [21] tells us that countable unions and countable intersections of analytic sets are analytic. Moreover, if $f : Y \rightarrow Z$ is a Borel mapping between two complete separable metric spaces, then images and preimages under f of analytic sets are analytic (also by Proposition 14.4 in [21]). By Theorem 21.10 in [21] (Luzin’s theorem), every analytic subset of a complete separable metric space Y is ν -measurable for every σ -finite Borel measure ν on Y .

When proving Theorem 1.4 we will need Proposition 1.5 with a lower semicontinuous ρ only, in which case the proof can be considerably simplified (in particular the use of Lemma 5.5 can be avoided). As we find it interesting that Proposition 1.5 is true also for Borel functions we give a proof of the more general result, for which we need the following result from Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [20, Lemma 2.4].

Lemma 5.5. *Let Z be a metric space and let \mathcal{Y} be a class of functions $\rho : Z \rightarrow [0, \infty]$ such that the following properties hold:*

- (a) \mathcal{Y} contains all continuous $\rho : Z \rightarrow [0, \infty]$;
- (b) if $\rho_j \in \mathcal{Y}$ and $\rho_j \nearrow \rho$ then $\rho \in \mathcal{Y}$;
- (c) if $\rho, \sigma \in \mathcal{Y}$ and $r, s \in \mathbf{R}^+$ then $r\rho + s\sigma \in \mathcal{Y}$;
- (d) if $\rho \in \mathcal{Y}$ and $0 \leq \rho \leq 1$ then $1 - \rho \in \mathcal{Y}$.

Then \mathcal{Y} contains all Borel functions $\rho : Z \rightarrow [0, \infty]$.

Proof of Proposition 1.5. For $\alpha > 0$, let \mathcal{L}_α consist of all continuous curves $\gamma : [0, 1] \rightarrow X$ with $\text{Lip } \gamma \leq \alpha$. (In this proof we want all curves parameterized on the same interval, and do *not* assume that they are parameterized by arc length.) Then \mathcal{L}_α is a metric space with respect to the supremum norm. Since X is complete, it

follows from Ascoli's theorem that \mathcal{L}_α is complete. (See e.g. Royden [27, p. 169] for a version of Ascoli's theorem valid for metric space valued equicontinuous functions.)

As X is separable, it is easily verified that \mathcal{L}_α is separable. Indeed, let \mathcal{U} be a countable base of the topology on X and let \mathcal{P} be the family of all finite sequences $Q = ((I_1, U_1), \dots, (I_m, U_m))$, where $I_i \subset [0, 1]$ are closed intervals with rational endpoints and U_i are selected from \mathcal{U} . Then \mathcal{P} is countable. For each $Q = ((I_1, U_1), \dots, (I_m, U_m)) \in \mathcal{P}$, let \mathcal{L}_Q be the family of all $\gamma \in \mathcal{L}_\alpha$ such that $\gamma(I_i) \subset U_i$, $i = 1, \dots, m$. Choosing one curve from every nonempty \mathcal{L}_Q provides us with a countable dense system in \mathcal{L}_α .

Let \mathcal{Y}_α be the collection of all $\rho : X \rightarrow [0, \infty]$ for which the functional $\Phi_\rho : \mathcal{L}_\alpha \rightarrow [0, \infty]$ defined by

$$\Phi_\rho : \gamma \mapsto \int_\gamma \rho ds$$

is Borel. Lemma 2.2 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [20] shows that if ρ is continuous then $\rho \in \mathcal{Y}_\alpha$. If $\rho_j \in \mathcal{Y}$ and $\rho_j \nearrow \rho$ then the monotone convergence theorem implies that for all $\gamma \in \mathcal{L}_\alpha$,

$$\Phi_{\rho_j}(\gamma) = \int_\gamma \rho_j ds \nearrow \int_\gamma \rho ds = \Phi_\rho(\gamma),$$

i.e. that Φ_ρ is a limit of Borel functions on \mathcal{L}_α , and hence Borel. Thus, $\rho \in \mathcal{Y}_\alpha$. In particular, (a) and (b) in Lemma 5.5 are satisfied by \mathcal{Y}_α . The properties (c) and (d) therein follow from the linearity of the integral.

We can thus conclude from Lemma 5.5 that for every $\alpha > 0$ and every Borel $\rho : X \rightarrow [0, \infty]$, the functional $\Phi_\rho : \mathcal{L}_\alpha \rightarrow [0, \infty]$ is Borel. Let $\rho : X \rightarrow [0, \infty]$ be a fixed Borel function and $\alpha > 0$ be arbitrary. We shall prove that the set $G = \{x \in X : u_\rho(x) < \alpha\}$ is measurable.

First, assume that $\rho \geq \delta$ for some $\delta > 0$. Note that if $x \in G$, then for every $\gamma \in \Gamma_x$ with $\int_\gamma \rho ds < \alpha$, we have $l_\gamma < \alpha/\delta$ and hence the reparameterized curve $\tilde{\gamma}(t) := \gamma(l_\gamma t)$ belongs to $\mathcal{L}_{\alpha/\delta}$. It follows that for $x \in G$, the infimum in the definition of u_ρ can equivalently be taken over all $\gamma \in \Gamma_x \cap \mathcal{L}_{\alpha/\delta}$. We define

$$\Gamma_1 = \left\{ \gamma \in \mathcal{L}_{\alpha/\delta} : \int_\gamma \rho ds < \alpha \right\} \quad \text{and} \quad \Gamma_2 = \{\gamma \in \mathcal{L}_{\alpha/\delta} : \gamma(1) \notin U\}.$$

Also let $f_j : \mathcal{L}_{\alpha/\delta} \rightarrow X$ be the evaluation maps given by $f_j(\gamma) = \gamma(j)$, $j = 0, 1$, which are clearly 1-Lipschitz, and thus Borel.

By the above, the functional $\Phi_\rho : \mathcal{L}_{\alpha/\delta} \rightarrow [0, \infty]$ is Borel, and thus $\Gamma_1 = \Phi_\rho^{-1}([0, \alpha))$ is a Borel subset of $\mathcal{L}_{\alpha/\delta}$. The set Γ_2 is the preimage of the analytic set $X \setminus U$ under the Borel mapping f_1 , and thus Γ_2 is analytic by Proposition 14.14 in Kechris [21]. It thus follows from Proposition 14.4 in [21] that $\Gamma_1 \cap \Gamma_2$ is analytic. Since $G = f_0(\Gamma_1 \cap \Gamma_2)$, we conclude that G is analytic, and thus measurable, by Luzin's theorem [21, Theorem 21.10].

Now, let $\rho : X \rightarrow [0, \infty]$ be arbitrary and set $\rho_j = \rho + 1/j$, $j = 1, 2, \dots$. By the above, each u_{ρ_j} is measurable. We shall show that $u_\rho = \lim_{j \rightarrow \infty} u_{\rho_j}$, which implies the measurability of u_ρ .

Given $x \in X$ and $\varepsilon > 0$, there exists $\gamma \in \Gamma_x$ such that $\int_\gamma \rho ds < u_\rho(x) + \varepsilon$. Since γ is rectifiable, we have

$$u_\rho(x) \leq u_{\rho_j}(x) \leq \int_\gamma \rho_j ds \leq \int_\gamma (\rho + 1/j) ds < u_\rho(x) + \varepsilon + \frac{l_\gamma}{j}.$$

Letting $j \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ shows that $u_\rho(x) = \lim_{j \rightarrow \infty} u_{\rho_j}(x)$ for all $x \in X$, and we conclude that u_ρ is measurable. \square

We are now ready to prove Theorem 1.4. It will be obtained in a more general form, using coanalytic sets. For separable X we have the following result.

Theorem 5.6. *Assume that X is complete and separable, and that every bounded $u \in N^{1,p}(X)$ is quasicontinuous. Then every coanalytic p -path open set $U \subset X$ is quasiopen.*

For nonseparable spaces we use the fact that $\text{supp } \mu$ is always separable, by Proposition 1.6 in [4]. This way we can avoid nonseparable analytic (Suslin) sets and reduce our considerations to separable spaces where Suslin and analytic sets are the same, cf. Hansell [15] and KeChris [21]. The following result is primarily designed for nonseparable spaces. However, it improves the criterion also for separable spaces, since we do not impose any assumptions on $U \setminus \text{supp } \mu$.

Theorem 5.7. *Assume that $\text{supp } \mu$ is complete, and that every bounded $u \in N^{1,p}(X)$ is quasicontinuous. Then every p -path open set $U \subset X$, such that $\text{supp } \mu \setminus U$ is analytic (in $\text{supp } \mu$), is quasiopen.*

Theorem 5.6 is a special case of Theorem 5.7, but we will prove Theorem 5.6 first and then use it to prove Theorem 5.7.

Remark 5.8. By Theorem 1.1 in Björn–Björn–Shanmugalingam [10] (or [4, Theorem 5.29]) and the comments after Proposition 4.7 in Björn–Björn–Lehrbäck [9], the assumptions in Theorem 5.7 hold in particular if X (or more generally $\text{supp } \mu$) is complete and locally compact, and continuous functions are dense in $N^{1,p}(X)$ or equivalently in $N^{1,p}(\text{supp } \mu)$. The last equivalence follows from Lemma 5.19 in [4], which also implies that every (bounded) function in $N^{1,p}(X)$ is quasicontinuous if and only if every (bounded) function in $N^{1,p}(\text{supp } \mu)$ is quasicontinuous.

At this point we would also like to mention that it follows from the recent results in Ambrosio–Colombo–Di Marino [2] and Ambrosio–Gigli–Savaré [3] that, if $\text{supp } \mu$ is a complete doubling metric space and $1 < p < \infty$, then Lipschitz functions are dense in $N^{1,p}(X)$. Thus, the assumptions on X in Theorem 5.7 hold if X (or $\text{supp } \mu$) is a complete doubling metric space and $1 < p < \infty$. It is easily verified that such a space is automatically locally compact.

Proof of Theorem 5.6. We follow the proof of Theorem 1.1. Let $\rho \in L^p(X)$ be as therein. The Vitali–Carathéodory theorem (Proposition 7.14 in Folland [12]) provides us with a lower semicontinuous pointwise majorant of ρ which also belongs to $L^p(X)$. We can therefore without loss of generality assume that ρ is lower semicontinuous.

The measurability of the function u in (4.1) is now guaranteed by Proposition 1.5 and the assumption that U is coanalytic, rather than by the p -Poincaré inequality and Theorem 1.11 in [20]. Also, since quasicontinuity of bounded Newtonian functions is assumed, it need not be concluded from the p -Poincaré inequality. The rest of the proof goes through verbatim. \square

Proof of Theorem 5.7. Let $Y = \text{supp } \mu$. It follows from Proposition 1.6 in [4] that Y is separable. Let $U \subset X$ be a p -path open set such that $Y \setminus U$ is analytic. It follows from Proposition 1.53 in [4] that p -almost no curve intersects $X \setminus Y$, and thus $U \cap Y$ is also p -path open.

Hence, as all bounded $u \in N^{1,p}(Y)$ are quasicontinuous by Lemma 5.19 in [4], Theorem 5.6 implies that $U \cap Y$ is quasiopen in Y , i.e. for every $\varepsilon > 0$ there is a relatively open set $G \subset Y$ such that $U \cap Y \subset G$ and $C_p^Y(G \setminus U) < \varepsilon$. Thus there exists $v \in N^{1,p}(Y)$ such that $v \geq \chi_{G \setminus U}$ and $\|v\|_{N^{1,p}(Y)}^p < \varepsilon$. Since p -almost no curve intersects $X \setminus Y$ and $\mu(X \setminus Y) = 0$, extending v by 1 to $X \setminus Y$ shows that $C_p(G \setminus U) \leq \|v\|_{N^{1,p}(X)}^p = \|v\|_{N^{1,p}(Y)}^p < \varepsilon$.

Let $G' = G \cup (X \setminus Y)$, which is open and contains U . By Proposition 1.53 in [4] we see that $C_p(X \setminus Y) = 0$ and hence

$$C_p(G' \setminus U) \leq C_p(G \setminus U) + C_p(X \setminus Y) = C_p(G \setminus U) < \varepsilon,$$

showing that U is quasiopen in X . □

Proof of Theorem 1.4. This is a special case of Theorem 5.7. □

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